COVERINGS AND COMPRESSED LATTICES

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Abstract. Motivated by "finite alphabet" approximation problems in infinite-dimensional Banach spaces we study the behavior of the inhomogeneous minimum of a convex body K with respect to the integral lattice $\mathbb{Z}^n$, if $\mathbb{Z}^n$ is compressed along (some of) the coordinate axes. In particular, we show that for certain convex bodies and deformations the inhomogeneous minimum can be arbitrarily large which answers a question in the negative posted in the context with the above mentioned approximation problems.

1. Introduction

In [1] the authors study several approximation properties related to the problem of approximating an element of an infinite-dimensional space by a discrete structure which might be regarded as a kind of infinite-dimensional lattice. Regarding these approximations they pose at the end of their article several questions and the corresponding finite-dimensional analogues [1, Questions 7.1, 7.2]. Here we investigate these finite-dimensional versions for which we need some basic notation from Geometry of Numbers (see, e.g., [3, 2]).

The set of all symmetric convex bodies with respect to the origin 0 in $\mathbb{R}^n$ with non-empty interior is denoted by $\mathcal{K}_0^n$. For $K \in \mathcal{K}_0^n$ the inhomogeneous minimum of K with respect to the integral lattice $\mathbb{Z}^n$ is defined as

$$\mu(K) = \min\{\mu > 0 : \mathbb{Z}^n + \mu\, K = \mathbb{R}^n\},$$

i.e., it is the smallest positive number such that the dilated body $\mu(K)\, K$ covers $\mathbb{R}^n$ by translates of the lattice $\mathbb{Z}^n$. Obviously, for any positive number $\rho > 0$ we have $\mu(\rho\, K) = (1/\rho)\mu(K)$, and the inhomogeneous minimum measures how well the space can be covered by lattice translates of K. According to its covering properties three families of convex bodies are considered in [1]:

$$\mathcal{C}_1^n = \{K \in \mathcal{K}_0^n : \mu(K) \leq 1\},$$
$$\mathcal{C}_2^n = \left\{K \in \mathcal{K}_0^n : \mu\big(\mathrm{diag}(\rho_1, \ldots, \rho_n)\, K\big) \leq 1, \text{for all } \rho_i \in [1, 2]\right\},$$
$$\mathcal{C}_3^n = \left\{K \in \mathcal{K}_0^n : [0, 1]^n \subseteq \{0, 1\}^n + K\right\}.$$

Here $\mathrm{diag}(\rho_1, \ldots, \rho_n)$ denotes the $n \times n$ diagonal matrix with diagonal entries ρ_i. Observe, that in the case $\rho_i = \rho$, $1 \leq i \leq n$, we obviously have $\mu\big(\mathrm{diag}(\rho_1, \ldots, \rho_n)\, K\big) = \mu(\rho K) = (1/\rho)\mu(K)$. The first set $\mathcal{C}_1^n$ consists just of

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all bodies which cover the space by lattice translates and it is also clear that the bodies in $\mathcal{C}_2^n$ and $\mathcal{C}_3^n$ share this property. However, the inclusions are strict, i.e.,

$$\mathcal{C}_2^n \subsetneq \mathcal{C}_1^n, \quad \mathcal{C}_3^n \subsetneq \mathcal{C}_1^n.$$

For instance, let $K = \operatorname{conv}\{\pm(1/2,1)^\intercal, \pm(1/2,2)^\intercal\} \in \mathcal{C}_1^n$ (see Figure 1 left). Clearly K is a lattice space filler, i.e., a body which covers the space by lattice translates in such a way that two different translates do not overlap, but apparently K is not contained in $\mathcal{C}_3^n$. Moreover in Section 4 we will show that $K \in \mathcal{C}_2^n$ and hence we also know $\mathcal{C}_2^n \not\subseteq \mathcal{C}_3^n$; in the figure (on the right) the parallelogram K has been multiplied by $\operatorname{diag}(6/5, 11/10)$.

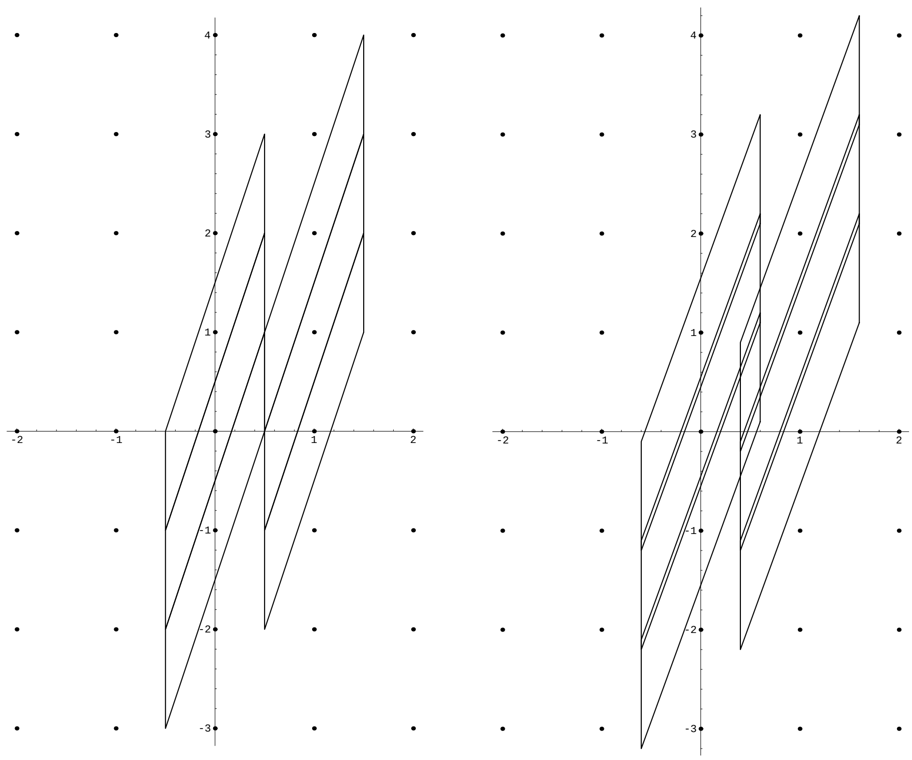

FIGURE 1. An example for $\mathcal{C}_1^n \not\subseteq \mathcal{C}_3^n$ and $\mathcal{C}_2^n \not\subseteq \mathcal{C}_3^n$.

In order to verify that $\mathcal{C}_1^n \not\subseteq \mathcal{C}_2^n$ we use the following example taken from [1]. Let K be the lattice space filler $\operatorname{conv}\{\pm(1/4,1)^\intercal, \pm(3/4,1)^\intercal\}$ (see Figure 2 left). If we multiply K by $\operatorname{diag}(10/9, 1)$ then we see (Figure 2 right) that it is not a covering anymore. Since $K \in \mathcal{C}_3^n$, the example also shows that $\mathcal{C}_3^n \not\subseteq \mathcal{C}_2^n$.

In [1, Question 7.3] the authors raised the question whether we can have $\mathcal{C}_1^n \subseteq \mathcal{C}_2^n$ at least "up to a constant", i.e.,

(1.1) Does there exist a universal constant $c \geq 1$ such that $c\,\mathcal{C}_1^n \subseteq \mathcal{C}_2^n$, i.e., $c\,K \in \mathcal{C}_2^n$ for all $K \in \mathcal{C}_1^n$?

We will answer that question in the negative in Section 2. In fact, we will show that there even does not exist a constant which might depend on the dimension.

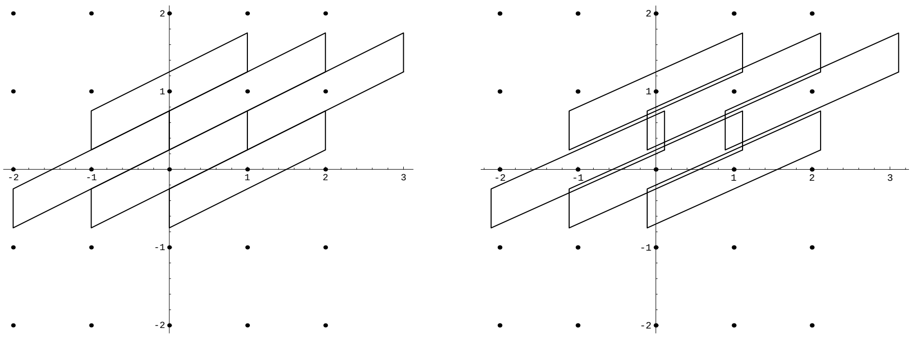

FIGURE 2. An example for $\mathcal{C}_1^n \not\subseteq \mathcal{C}_2^n$ and $\mathcal{C}_3^n \not\subseteq \mathcal{C}_2^n$.

Theorem 1.1. *For any $n, M \in \mathbb{N}$, $n \geq 2$, there exists a convex body $K \in \mathcal{C}_1^n$ such that $MK \notin \mathcal{C}_2^n$.*

Hence in order to belong to $\mathcal{C}_2^n$ or $\mathcal{C}_3^n$ a body $K \in \mathcal{C}_1^n$ has to satisfy more structural properties. In order to describe such a property which was suggested in [1] we introduce the following notation: for a subset $I \subseteq \{1, \dots, n\}$ let L_I be the $(\#I)$-dimensional coordinate plane given by

$$L_I = \big\{x \in \mathbb{R}^n : x_i = 0 \text{ for all } i \in \{1, \dots, n\} \setminus I\big\},$$

where $\#I$ denotes the cardinal of I and $L_{\{1,\dots,n\}}$ is meant to be $\mathbb{R}^n$. With this notation we set

$$\overline{\mathcal{C}}_1^n = \big\{K \in \mathcal{K}_0^n : \mu(K \cap L_I) \leq 1 \text{ for all } I \subseteq \{1, \dots, n\}\big\},$$

where the inhomogeneous minimum $\mu(K \cap L_I)$ is taken with respect to the $(\#I)$-dimensional integral lattice $\mathbb{Z}^n \cap L_I$. Regarding this restricted family $\overline{\mathcal{C}}_1^n$, Dilworth et. al. asked [1, Questions 7.1/7.2]:

(1.2)
- I) Is $\overline{\mathcal{C}}_1^n \subseteq \mathcal{C}_2^n$ or $\overline{\mathcal{C}}_1^n \subseteq \mathcal{C}_3^n$?
- II) Does there exist (at least) a universal constant $c \geq 1$ such that $c\,\overline{\mathcal{C}}_1^n \subseteq \mathcal{C}_2^n$ or $c\,\overline{\mathcal{C}}_1^n \subseteq \mathcal{C}_3^n$?

Unfortunately, we can settle that problem only in the planar case, where I) has an affirmative answer and which can be embedded in the following slightly more general result.

Theorem 1.2. *Let $n \geq 2$. Then $\lceil n/2 \rceil\, \overline{\mathcal{C}}_1^n \subseteq \mathcal{C}_2^n$ and $(n/2)\, \overline{\mathcal{C}}_1^n \subseteq \mathcal{C}_3^n$.*

Moreover, as shown in Remark 3.2, these inclusions are already strict in the case $n = 2$.

The paper is organized as follows. The proof of Theorem 1.1 will be given in the next section. In Section 3 we will give our partial answer to question (1.2), and final remarks and comments are contained in Section 4.

2. A negative answer to question (1.1)

Obviously, if a convex body $K \in \mathcal{K}_0^n$ contains the cube C_n of edge length 1 centered at the origin then $\mu(K) \leq 1$. Since C_n is symmetric with respect to all coordinate hyperplanes we certainly have that $C_n \subseteq \operatorname{diag}(\rho_1, \dots, \rho_n)\, C_n$ for any choice of real numbers $\rho_i \geq 1$. Thus if

$$\mathrm{r}(K; C_n) = \max\{r > 0 : r\, C_n \subseteq K\}$$

denotes the inradius of K with respect to C_n, we have for all $K \in \mathcal{K}_0^n$

$$\frac{1}{\mathrm{r}(K; C_n)}\, K \in \mathcal{C}_2^n.$$

Of course, even for bodies in the class $\mathcal{C}_1^n$, the factor $1/\mathrm{r}(K; C_n)$ might be arbitrarily large. The next lemma, however, shows that this is all what we actually can expect, i.e., $1/\mathrm{r}(K; C_n)$ has the right order for a factor c guaranteeing $c\, K \in \mathcal{C}_2^n$ for all $K \in \mathcal{C}_1^n$.

Lemma 2.1. *For any dimension $n \geq 2$ and any positive integer M there exists a lattice space filler $P_M^n \in \mathcal{K}_0^n$ with $\mathrm{r}(P_M^n; C_n) = 1/\big(2(M+1)\big)$ and $c\, P_M^n \in \mathcal{C}_2^n$ only if $c \geq M$.*

Proof. We start in dimension $n = 2$. For the given integer M we consider the parallelogram $P_M^2 = \operatorname{conv}\{\pm(1/2)(M+1, M+2)^\intercal, \pm(1/2)(M-1, M)^\intercal\}$ (see Figure 3 left).

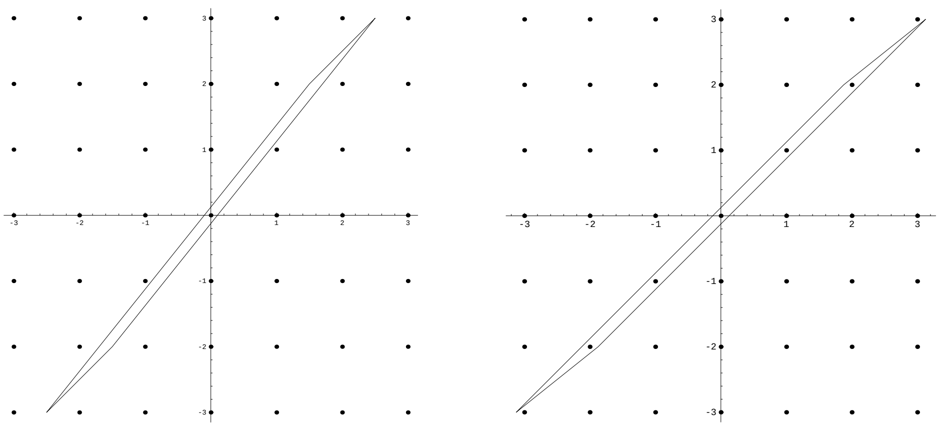

FIGURE 3. The parallelograms P_M^2 and $\overline{P}_M^2$ for $M = 4$.

Clearly P_M^2 is the linear image of the cube C_2 under the unimodular transformation

$$A = \begin{pmatrix} M & 1 \\ M+1 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}).$$

Hence P_M^2 is a lattice space filler (see Figure 4 left) and the norm $|\cdot|_{P_M^2}$ associated to P_M^2 is given by

$$|(x,y)^\intercal|_{P_M^2} = 2 \max\left\{|-x+y|, \big|(M+1)\,x - M\,y\big|\right\}.$$

From that we also conclude that $\mathrm{r}(P_M^2; C_2) = 1/\big(2(M+1)\big)$.

Multiplying P_M^2 by the diagonal matrix $\mathrm{diag}\big((M+1)/M,1\big)$ leads to a parallelogram $\overline{P}_M^2$ (see Figure 3 right) with norm

$$|(x,y)^\intercal|_{\overline{P}_M^2} = 2\,\max\left\{\left|-\frac{M}{M+1}x+y\right|,|M\,x-M\,y|\right\}.$$

In order to determine the inhomogeneous minimum of $\overline{P}_M^2$ we note that the inhomogeneous minimum of a convex body $K\in\mathcal{K}_0^n$ is the maximum distance which a point can have from the lattice $\mathbb{Z}^n$, where the distance is measured with respect to the norm associated to the body K (see [3, pp. 98–99]). Hence

$$\mu(\overline{P}_M^2) = \max_{(x,y)^\intercal\in\mathbb{R}^2}\ \min_{(z_1,z_2)^\intercal\in\mathbb{Z}^2}\left|\begin{pmatrix}z_1\\z_2\end{pmatrix}-\begin{pmatrix}x\\y\end{pmatrix}\right|_{\overline{P}_M^2} \geq \min_{(z_1,z_2)^\intercal\in\mathbb{Z}^2}\left|\begin{pmatrix}z_1\\z_2\end{pmatrix}-\begin{pmatrix}0\\\frac12\end{pmatrix}\right|_{\overline{P}_M^2} = M$$

(see Figure 4 right). In fact, it is easy to see that we have actually equality. Hence for any constant c with $c\,P_M^2\in\mathcal{C}_2^n$ we have the lower bound $c\geq M$ which proves the lemma in the planar case.

For general $n\geq 3$ we just take a parallelepiped given as an iterated prism over P_M^2, i.e.,

$$P_M^n := P_M^2 + \mathrm{conv}\left\{-\frac12 e_3,\frac12 e_3\right\}+\ldots+\mathrm{conv}\left\{-\frac12 e_n,\frac12 e_n\right\}.$$

Here e_i denotes the i-th canonical unit vector, and P_M^2 is embedded in the plane $L_{\{1,2\}}$. Of course, P_M^n is a lattice space filler with $\mathrm{r}(P_M^n;C_n)=\mathrm{r}(P_M^2;C_2)$, and multiplying P_M^n by the diagonal matrix $\mathrm{diag}\big((M+1)/M,1,1,\ldots,1\big)$ leads to the parallelepiped

$$\overline{P}_M^n = \overline{P}_M^2 + \mathrm{conv}\left\{-\frac12 e_3,\frac12 e_3\right\}+\ldots+\mathrm{conv}\left\{-\frac12 e_n,\frac12 e_n\right\}.$$

Hence $\mu(\overline{P}_M^n)=\mu(\overline{P}_M^2)\geq M$. □

Figure 4 shows that the parallelogram P_4^2 is clearly a space filler, whereas $\overline{P}_4^2$ needs to be multiplied by 4 in order to cover the plane by translates of $\mathbb{Z}^2$.

Of course, Theorem 1.1 is a direct consequence of the above result.

3. Partial answers to question (1.2)

The proof of Theorem 1.2 relies heavily on a theorem about planar finite lattice coverings (see [4]). In order to state it we need the notion of lattice polygon, which is the convex hull of finitely many lattice points in the plane. Moreover, for such a lattice polygon P, the boundary of P is denoted by $\mathrm{bd}\,P$.

Theorem 3.1 ([4]). *Let $K\in\mathcal{C}_1^2$ and let $P\subset\mathbb{R}^2$ be a lattice polygon. Then $P\subseteq(P\cap\mathbb{Z}^2)+K$ if and only if $\mathrm{bd}\,P\subseteq(P\cap\mathbb{Z}^2)+K$.*

Proof of Theorem 1.2. On account of Theorem 3.1 both inclusions can be easily proved by inductive arguments.

First we treat the inclusion $(n/2)\,\overline{\mathcal{C}}_1^n\subseteq\mathcal{C}_3^n$. Let $n=2$ and let $K\in\overline{\mathcal{C}}_1^2$. Then $K\in\mathcal{C}_1^2$ and moreover $K\cap L_{\{i\}}$ covers the coordinate axis $L_{\{i\}}$ by translates of

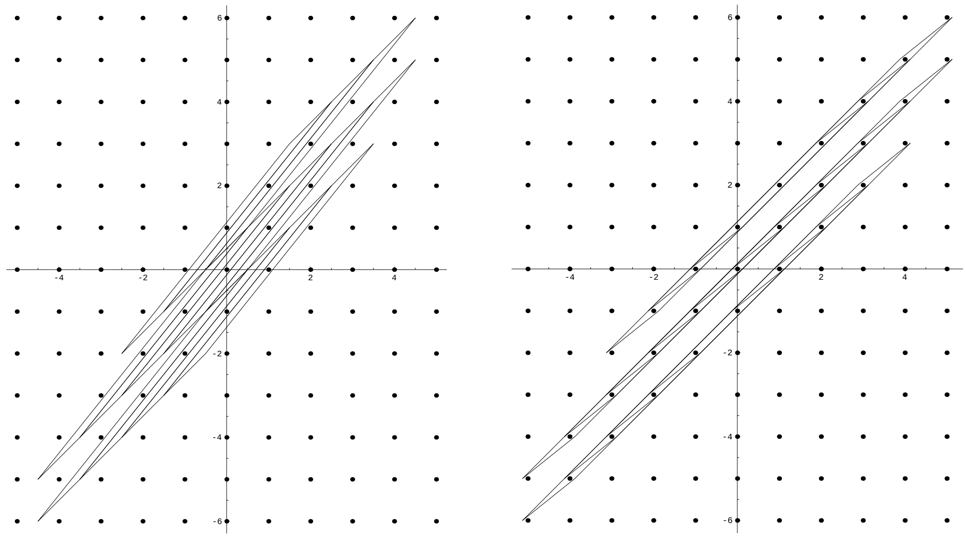

FIGURE 4. The parallelogram $\overline{P}_4^2$ is not a lattice space filler.

$\mathbb{Z}^2 \cap L_{\{i\}}$, $i = 1, 2$. In particular, the length of the sections $K \cap L_{\{i\}}$, $i = 1, 2$, are not smaller than 1, which shows that

$$\operatorname{conv}\{0, e_i\} \subseteq \left[\left(K \cap L_{\{i\}}\right) + 0\right] \cup \left[\left(K \cap L_{\{i\}}\right) + e_i\right], \quad i = 1, 2.$$

Therefore the boundary of the square $[0,1]^2$ is covered by $\{0,1\}^2 + K$, and Theorem 3.1 implies that $[0,1]^2 \subseteq \{0,1\}^2 + K$. Thus $K \in \mathcal{C}_3^2$, as required.

Next let $n \geq 3$ and let $K \in \overline{\mathcal{C}}_1^n$. We have to show that for any $x \in [0,1]^n$ there exists $v \in \{0,1\}^n$ such that $|x - v|_K \leq n/2$, where again $|\cdot|_K$ denotes the norm induced by K. So let $x = (x_1, \ldots, x_n) \in [0,1]^n$ and by our hypothesis we may assume $0 < x_i < 1$, $1 \leq i \leq n$. Since $K + \mathbb{Z}^n$ is a covering, there exists $b \in \mathbb{Z}^n$ verifying $|x - b|_K \leq 1$. Now if $b \notin \{0,1\}^n$ then there exists $\overline{\lambda}$, with $0 < \overline{\lambda} \leq 1/2$, such that $(1 - \overline{\lambda})x + \overline{\lambda}b$ is contained in a facet F of the cube $[0,1]^n$. Thus, by induction, we may assume that there exists $v \in \{0,1\}^n \cap F$ such that

$$\left|(1 - \overline{\lambda})x + \overline{\lambda}b - v\right|_K \leq \left|(1 - \overline{\lambda})x + \overline{\lambda}b - v\right|_{K \cap (\operatorname{aff} F - v)} \leq \frac{n-1}{2};$$

here $\operatorname{aff} F$ denotes the affine hull of F. Finally, by the triangle inequality we can conclude that

$$|x - v|_K \leq \frac{n-1}{2} + \overline{\lambda}|x - b|_K \leq \frac{n}{2}.$$

Next we consider the inclusion $\overline{\mathcal{C}}_1^n \subseteq \lceil n/2 \rceil\, \mathcal{C}_2^n$. Let $n = 2$ and let $\rho_1, \rho_2 \geq 1$. For $K \in \overline{\mathcal{C}}_1^2$ we certainly know that also the body $\operatorname{diag}(\rho_1, \rho_2)\, K \cap L_{\{i\}}$ covers the coordinate axis $L_{\{i\}}$. Hence, as in the first case, we conclude

$$\operatorname{diag}(\rho_1, \rho_2)\, K \in \mathcal{C}_3^2. \tag{3.1}$$

Thus $\mu(\operatorname{diag}(\rho_1, \rho_2)\, K) \leq 1$ and, in particular, $K \in \mathcal{C}_2^2$.

Now let $n \geq 3$. Let $K \in \overline{\mathcal{C}}_1^n$ and let $\rho_i \geq 1$, $1 \leq i \leq n$. Furthermore, let $x \in [0,1]^n$, and for $I \subseteq \{1,\ldots,n\}$ we write x_I to denote the orthogonal projection of x onto the $(\#I)$-dimensional coordinate plane L_I. By (3.1) we know that for any subset I with $\#I = 2$ there exists $v_I \in L_I \cap \{0,1\}^n$ such that

$$x_I \in v_I + \left[\operatorname{diag}(\rho_1,\ldots,\rho_n)\, K \cap L_I\right].$$

Since $K \in \overline{\mathcal{C}}_1^n$ we have an analogous relation for singletons I. Thus if n is odd we get

$$\begin{aligned} x &= x_{\{1,2\}} + x_{\{3,4\}} + \cdots + x_{\{n-2,n-1\}} + x_{\{n\}} \\ &\in v_{\{1,2\}} + v_{\{3,4\}} + \cdots + v_{\{n-2,n-1\}} + v_{\{n\}} + \left\lceil \frac{n}{2} \right\rceil \operatorname{diag}(\rho_1,\ldots,\rho_n)\, K, \end{aligned}$$

whereas for n even, since we can decompose the space in an orthogonal sum of only 2-dimensional spaces, we obtain

$$\begin{aligned} x &= x_{\{1,2\}} + x_{\{3,4\}} + \cdots + x_{\{n-1,n\}} \\ &\in v_{\{1,2\}} + v_{\{3,4\}} + \cdots + v_{\{n-1,n\}} + \frac{n}{2} \operatorname{diag}(\rho_1,\ldots,\rho_n)\, K. \end{aligned}$$

This shows that $\lceil n/2 \rceil \operatorname{diag}(\rho_1,\ldots,\rho_n)\, K \in \overline{\mathcal{C}}_3^n$, and so we get $\lceil n/2 \rceil\, K \in \mathcal{C}_2^n$. $\square$

Remark 3.2. *The parallelogram*

$$P_\varepsilon = \operatorname{conv}\left\{ \pm(\varepsilon,0)^\intercal, \pm\left(\frac{\varepsilon}{4\varepsilon-1}, \frac{2\varepsilon}{4\varepsilon-1}\right)^\intercal \right\}, \quad \frac{1}{4} < \varepsilon < \frac{1}{2},$$

as depicted in Figure 5, shows that the inclusion $\overline{\mathcal{C}}_1^2 \subsetneq \mathcal{C}_3^2$ is strict: it clearly covers the square by translates of $\{0,1\}^2$ but $\mu(P_\varepsilon \cap L_{\{1\}}) > 1$.

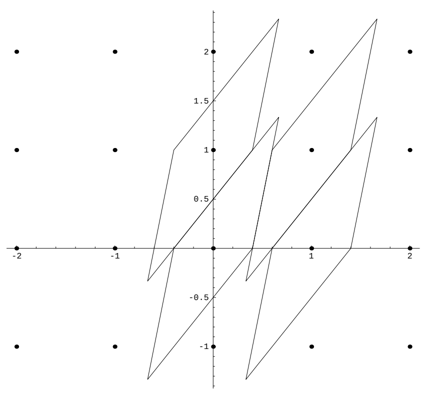

FIGURE 5. The parallelogram $P_\varepsilon \in \mathcal{C}_3^2 \backslash \overline{\mathcal{C}}_1^2$ for $\varepsilon = 2/5$.

The same example (as well as the parallelogram shown in Figure 1) shows that the inclusion $\overline{\mathcal{C}}_1^2 \subsetneq \mathcal{C}_2^2$ is also strict.

4. Final remarks

In order to complete the study of the possible inclusions among the different families, we still have to show $\mathcal{C}_2^n \not\subseteq \mathcal{C}_3^n$, as promised in the introduction. To this end we consider again the parallelogram $K = \operatorname{conv}\{\pm(1/2,1)^\intercal, \pm(1/2,2)^\intercal\}$ (see Figure 1 left). Clearly K is a lattice space filler, and moreover, it is easy to check that K covers the slab $\big\{(x,y)^\intercal \in \mathbb{R}^2 : -1/2 \le x \le 1/2\big\}$ by translates of $\mathbb{Z}^2 \cap \{x=0\}$, i.e.,

$$\{k\,e_2 : k \in \mathbb{Z}\} + K = \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : -\frac{1}{2} \le x \le \frac{1}{2}\right\}. \tag{4.1}$$

This is certainly still true if we replace K by $\operatorname{diag}(\rho_1,\rho_2)\,K$, as long as $\rho_1,\rho_2 \ge 1$.

So we have shown that all possible inclusions are strict.

We would like to remark that it seems to be rather difficult to give good bounds on the inhomogeneous minimum $\mu\big(\operatorname{diag}(\rho_1,\dots,\rho_n)\,K\big)$ depending on ρ_i. Regarded as a function in ρ_i, $\mu\big(\operatorname{diag}(\rho_1,\dots,\rho_n)\,K\big)$ "looks" to be rather "wild". For instance, Figure 6 shows a plot of the function $\mu\left(\operatorname{diag}(\rho,1)\,P_4^2\right)$, $1 \le \rho \le 4$. Here P_4^2 is the parallelogram described in the proof of Lemma 2.1, see Figure 3. Of course, whenever ρ is an integer we always have $\mu\left(\operatorname{diag}(\rho,1)\,P_4^2\right) \le 1$, but not much more can be said.

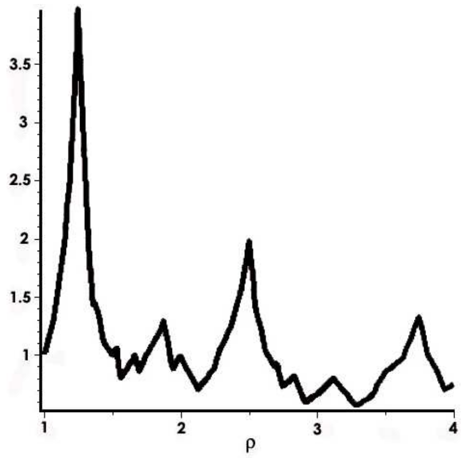

Figure 6. $\mu\left(\operatorname{diag}(\rho,1)\,P_4^2\right)$, $1 \le \rho \le 4$.

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